ESc 101: Fundamentals of Computing

Lecture 29

Mar 22, 2010

Given a square $n \times n$ matrix A, compute its inverse.

- The inverse exists if and only if the determinant of A is non-zero.
- Such an A is called invertible matrix.
- Inverse can be used to solve a system of linear equations.

Computing Inverse

• By definition, inverse of A is a matrix A^{-1} such that

 $A \cdot A^{-1} = I$

where *I* is the identity matrix:

$$I = egin{bmatrix} 1 & 0 & \dots & 0 \ 0 & 1 & \dots & 0 \ dots & dots & \ddots & dots \ 0 & 0 & \dots & 1 \end{bmatrix}.$$

- Taking entries of A^{-1} as unknowns, this gives rise to a system of n^2 linear equations, which has a unique solution, if it exists.
- This system can be solved using Gaussian elimination.
- However, there is a better way.

LUP DECOMPOSITION

- A lower triangular matrix *L* is a square matrix such that all entries above the diagonal are zero.
- An upper triangular matrix *U* is a square matrix such that all entries below the diagonal are zero.
- A permutation matrix *P* is a square matrix such that every row and column has exactly one entry 1 and all other entries 0.

THEOREM

Let A be an invertible matrix. There exists matrices L, U, P such that $A = L \cdot U \cdot P$, where L is a lower triangular matrix, U an upper triangular matrix, and P a permutation matrix.

Computing Inverse of a Lower Triangular Matrix

• Let *L* be a lower triangular matrix:

$$L = \begin{bmatrix} a_{0,0} & 0 & \dots & 0 \\ a_{1,0} & a_{1,1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,0} & a_{n-1,1} & \dots & a_{n-1,n-1} \end{bmatrix}.$$

Computing Inverse of a Lower Triangular Matrix

• Let:

$$L^{-1} = \begin{bmatrix} x_{0,0} & 0 & \dots & 0 \\ x_{1,0} & x_{1,1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1,0} & x_{n-1,1} & \dots & x_{n-1,n-1} \end{bmatrix}.$$

• Then, for every $0 \le i, k \le n-1$:

$$\sum_{j=k}^{i} x_{i,j} a_{j,k} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

Computing Inverse of a Lower Triangular Matrix

This immediately gives an algorithm to compute L^{-1} :

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    Start with i = 0;
    Set k = i, and solve for x[i][i]; // x[i][i] = 1/ a[i][i]
    Set k = k-1 until 0, and solve for x[i][k];
    Set i = i+1, and go to 2.
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Computing Inverse of an Upper Triangular and Permutation Matrix

- This is very similar to computing inverse of a lower triangular matrix.
- The inverse is also an upper triangular matrix.
- The inverse of a permutation matrix P is P^T , the transpose of P!

Computing Inverse of A

• Therefore, the inverse of $A = L \cdot U \cdot P$ is:

$$A^{-1} = P^T \cdot U^{-1} \cdot L^{-1}.$$

• The LUP decomposition also gives the determinant of A:

 $\det A = \det L \cdot \det U \cdot \det P.$

- The determinants of upper and lower triangular matrices are simply the products of diagonals.
- The determinant of permutation matrix equals 1 or -1 and can be easily calculated.

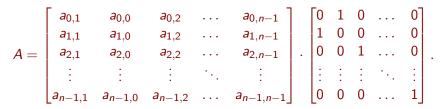
• Let $A = \begin{bmatrix} a_{0,0} & a_{0,1} & \dots & a_{0,n-1} \\ a_{1,0} & a_{1,1} & \dots & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,0} & a_{n-1,1} & \dots & a_{n-1,n-1} \end{bmatrix}.$

• If $a_{0,0} = 0$, then swap first column with *i*th column such that $a_{0,i} \neq 0$.

• If there is no such *i* then the determinant is zero and the matrix is not invertible.

• Suppose $a_{0,0} = 0$ and $a_{0,1} \neq 0$.

Then



 The first matrix, say *B*, has top left element non-zero and the second matrix, say *P*, is a permutation matrix.

• We now express *B* as a product of a lower triangular matrix and a matrix whose first column is all zero except the first entry.

Let

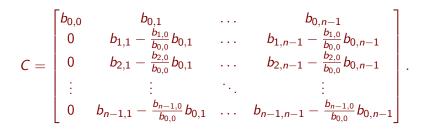
$$B = \begin{bmatrix} b_{0,0} & b_{0,1} & \dots & b_{0,n-1} \\ b_{1,0} & b_{1,1} & \dots & b_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-1,0} & b_{n-1,1} & \dots & b_{n-1,n-1} \end{bmatrix}$$

with $b_{0,0} \neq 0$.

Let

$$\tilde{L} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{b_{1,0}}{b_{0,0}} & 1 & 0 & \dots & 0 \\ \frac{b_{2,0}}{b_{0,0}} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{b_{n-1,0}}{b_{0,0}} & 0 & 0 & \dots & 1 \end{bmatrix}.$$

And



Then

 $B=\tilde{L}\cdot C.$

• Let A' be the matrix obtained by deleting first row and column of C:

$$C = \begin{bmatrix} b_{0,0} & b_{0,1} & \dots & b_{0,n-1} \\ 0 & & \\ \vdots & & A' & \\ 0 & & & \end{bmatrix}.$$

A' is invertible ⇔ C is invertible ⇔ B is invertible ⇔ A is invertible.
Now recursively decompose A' as: A' = L' · U' · P'.

• We can write C as:

$$C = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & L' & \\ 0 & & & \end{bmatrix} \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & \dots & b_{0,n-1} \\ 0 & & & \\ \vdots & U'P' \\ 0 & & & \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & L' & \\ 0 & & & & \end{bmatrix} \cdot \begin{bmatrix} b_{0,0} & b'_{0,1} & \dots & b'_{0,n-1} \\ 0 & & & \\ \vdots & U' & \\ 0 & & & & \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & \\ \vdots & P' \\ 0 & & & \end{bmatrix}$$

where $[b_{0,1} \cdots b_{0,n-1}] = [b'_{0,1} \cdots b'_{0,n-1}] \cdot P'$.

• Therefore, C gets decomposed as:

 $C = \hat{L} \cdot \hat{U} \cdot \hat{P},$

where \hat{L} is lower triangular matrix, \hat{U} is an upper triangular matrix, and \hat{P} is a permutation matrix.

• We already have:

 $A = \tilde{L} \cdot C \cdot \tilde{P}.$

• Hence:

 $A = \tilde{L} \cdot \hat{L} \cdot \hat{U} \cdot \hat{P} \cdot \tilde{P}.$

- It is easy to see that product of two lower triangular matrices is also lower triangular. Similarly for upper triangular and permutation matrices.
- Therefore:

 $A = L \cdot U \cdot P,$

where $L = \tilde{L} \cdot \hat{L}$ is a lower triangular matrix, $U = \hat{U}$ is an upper triangular matrix, $P = \hat{P} \cdot \tilde{P}$ is a permutation matrix.

• In addition, the diagonal entries in *L* are all 1's: follows from the above construction!