ESc 101: Fundamentals of Computing

Lecture 29

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The Problem

Given a square $n \times n$ matrix A, compute its inverse.

- The inverse exists if and only if the determinant of A is non-zero.
- Such an A is called invertible matrix.
- Inverse can be used to solve a system of linear equations.

Computing Inverse

- By definition, inverse of A is a matrix A^{-1} such that

$$A \cdot A^{-1} = I$$

  where I is the identity matrix:

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

- Taking entries of A^{-1} as unknowns, this gives rise to a system of n^2 linear equations, which has a unique solution, if it exists.
- This system can be solved using Gaussian elimination.
- However, there is a better way.

LUP Decomposition

- A lower triangular matrix L is a square matrix such that all entries above the diagonal are zero.
- An upper triangular matrix U is a square matrix such that all entries below the diagonal are zero.
- A permutation matrix P is a square matrix such that every row and column has exactly one entry 1 and all other entries 0.

Theorem

Let A be an invertible matrix. There exists matrices L, U, P such that $A = L \cdot U \cdot P$, where L is a lower triangular matrix, U an upper triangular matrix, and P a permutation matrix.

Computing Inverse of a Lower Triangular Matrix

- Let L be a lower triangular matrix:

$$L = \begin{bmatrix} a_{0,0} & 0 & \dots & 0 \\ a_{1,0} & a_{1,1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,0} & a_{n-1,1} & \dots & a_{n-1,n-1} \end{bmatrix}.$$

Computing Inverse of a Lower Triangular Matrix

- Let:

$$L^{-1} = \begin{bmatrix} x_{0,0} & 0 & \dots & 0 \\ x_{1,0} & x_{1,1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1,0} & x_{n-1,1} & \dots & x_{n-1,n-1} \end{bmatrix}.$$

- Then, for every $0 \leq i, k \leq n-1$:

$$\sum_{j=k}^{i} x_{i,j} a_{j,k} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

Computing Inverse of a Lower Triangular Matrix

This immediately gives an algorithm to compute L^{-1}:

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1. Start with i = 0;
2. Set k = i, and solve for x[i][i]; // x[i][i] = 1/ a[i][i]
3. Set k = k-1 until 0, and solve for x[i][k];
4. Set i = i+1, and go to 2.
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Computing Inverse of an Upper Triangular and Permutation Matrix

- This is very similar to computing inverse of a lower triangular matrix.
- The inverse is also an upper triangular matrix.
- The inverse of a permutation matrix P is P^T, the transpose of P!

Computing Inverse of A

- Therefore, the inverse of $A = L \cdot U \cdot P$ is:

$$A^{-1} = P^T \cdot U^{-1} \cdot L^{-1}.$$

- The LUP decomposition also gives the determinant of A:

$$\det A = \det L \cdot \det U \cdot \det P.$$

- The determinants of upper and lower triangular matrices are simply the products of diagonals.
- The determinant of permutation matrix equals 1 or -1 and can be easily calculated.

Doing LUP Decomposition

- Let

$$A = \begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,n-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,n-1} \end{bmatrix}.$$

- If $a_{0,0} = 0$, then swap first column with ith column such that $a_{0,i} \neq 0$.
- If there is no such i then the determinant is zero and the matrix is not invertible.

Doing LUP Decomposition

- Suppose $a_{0,0} = 0$ and $a_{0,1} \neq 0$.
- Then

$$A = \begin{bmatrix} a_{0,1} & a_{0,0} & a_{0,2} & \dots & a_{0,n-1} \\ a_{1,1} & a_{1,0} & a_{1,2} & \dots & a_{1,n-1} \\ a_{2,1} & a_{2,0} & a_{2,2} & \dots & a_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,0} & a_{n-1,2} & \dots & a_{n-1,n-1} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

- The first matrix, say B, has top left element non-zero and the second matrix, say $\tilde{P}$, is a permutation matrix.

Doing LUP Decomposition

- We now express B as a product of a lower triangular matrix and a matrix whose first column is all zero except the first entry.
- Let
$$B = \begin{bmatrix} b_{0,0} & b_{0,1} & \dots & b_{0,n-1} \\ b_{1,0} & b_{1,1} & \dots & b_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-1,0} & b_{n-1,1} & \dots & b_{n-1,n-1} \end{bmatrix}$$
with $b_{0,0} \neq 0$.
- Let
$$\tilde{L} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{b_{1,0}}{b_{0,0}} & 1 & 0 & \dots & 0 \\ \frac{b_{2,0}}{b_{0,0}} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{b_{n-1,0}}{b_{0,0}} & 0 & 0 & \dots & 1 \end{bmatrix}.$$

Doing LUP Decomposition

- And

$$C = \begin{bmatrix} b_{0,0} & b_{0,1} & \cdots & b_{0,n-1} \\ 0 & b_{1,1} - \frac{b_{1,0}}{b_{0,0}} b_{0,1} & \cdots & b_{1,n-1} - \frac{b_{1,0}}{b_{0,0}} b_{0,n-1} \\ 0 & b_{2,1} - \frac{b_{2,0}}{b_{0,0}} b_{0,1} & \cdots & b_{2,n-1} - \frac{b_{2,0}}{b_{0,0}} b_{0,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n-1,1} - \frac{b_{n-1,0}}{b_{0,0}} b_{0,1} & \cdots & b_{n-1,n-1} - \frac{b_{n-1,0}}{b_{0,0}} b_{0,n-1} \end{bmatrix}.$$

- Then

$$B = \tilde{L} \cdot C.$$

Doing LUP Decomposition

- Let A' be the matrix obtained by deleting first row and column of C:

$$C = \begin{bmatrix} b_{0,0} & b_{0,1} & \cdots & b_{0,n-1} \\ 0 & & & \\ \vdots & & \text{A'} & \\ 0 & & & \end{bmatrix}.$$

- A' is invertible $\Leftrightarrow$ C is invertible $\Leftrightarrow$ B is invertible $\Leftrightarrow$ A is invertible.
- Now recursively decompose A' as: $A' = L' \cdot U' \cdot P'$.

Doing LUP Decomposition

- We can write C as:

$$
\begin{aligned}
C &= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & L' & \\ 0 & & & \end{bmatrix} \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & \dots & b_{0,n-1} \\ 0 & & & \\ \vdots & & U'P' & \\ 0 & & & \end{bmatrix} \cdot \\
&= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & L' & \\ 0 & & & \end{bmatrix} \cdot \begin{bmatrix} b_{0,0} & b'_{0,1} & \dots & b'_{0,n-1} \\ 0 & & & \\ \vdots & & U' & \\ 0 & & & \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & P' & \\ 0 & & & \end{bmatrix}
\end{aligned}
$$

where $[b_{0,1} \cdots b_{0,n-1}] = [b'_{0,1} \cdots b'_{0,n-1}] \cdot P'$.

Doing LUP Decomposition

- Therefore, C gets decomposed as:

$$C = \hat{L} \cdot \hat{U} \cdot \hat{P},$$

where $\hat{L}$ is lower triangular matrix, $\hat{U}$ is an upper triangular matrix, and $\hat{P}$ is a permutation matrix.

- We already have:

$$A = \tilde{L} \cdot C \cdot \tilde{P}.$$

- Hence:

$$A \;=\; \tilde{L} \cdot \hat{L} \cdot \hat{U} \cdot \hat{P} \cdot \tilde{P}.$$

Doing LUP Decomposition

- It is easy to see that product of two lower triangular matrices is also lower triangular. Similarly for upper triangular and permutation matrices.
- Therefore:

  $$A = L \cdot U \cdot P,$$

  where $L = \tilde{L} \cdot \hat{L}$ is a lower triangular matrix, $U = \hat{U}$ is an upper triangular matrix, $P = \hat{P} \cdot \tilde{P}$ is a permutation matrix.
- In addition, the diagonal entries in L are all 1's: follows from the above construction!